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ON THE THEORY OF TRADE OFF AND ITS APPLICATION TO THE CONSTRUCT--ETC(U)
AUG 77 A HEDAYAT, S R LI AFOSR-76-3050

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20. Abstract

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On the Theory of Trade Off and its Application to the Construction of BIB Designs with Variable Support Sizes

by

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Abstract

A BIB design with b blocks is said to have the support size b^* when exactly b^* of the b blocks are distinct. BIB designs with $b^* < b$ have interesting applications in design of experiments and finite population sampling as explained in detail in Foody and Hedayat (1977). A method called "trade off" is introduced for the construction of BIB designs. We apply this method and some techniques in combinatorial topology to study BIB designs with arbitrary v and $k = 3$ in general and with $v = 7$ and $k = 3$ in particular. We determine the existence or nonexistence of BIB designs with any given b and b^* except the case when $b^* = 16$ for the family of BIB designs with $v = 7$ and $k = 3$.

1. Motivation. Suppose an experimenter wants to test and evaluate $v = 7$ treatments based on b blocks each of size $k = 3$. According to the usual homoscedastic linear additive model for measurements, the best possible design under any reasonable statistical criterion is a balanced incomplete block design (abbr. BIB design). This is a result due to Kiefer (1958, 1975). When b is not a multiple of 7, no BIB design exists and therefore the existing literature is not of much help to the experimenter. But if b is a multiple of 7, the designs do exist. Thus label the treatments as $1,2,\ldots,7$. For $b = 7$, one example of BIB design is

$$\begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & 4 & 6 \\ 4 & 5 & 7 \end{array} \qquad\qquad \begin{array}{ccc} 5 & 6 & 1 \\ 6 & 7 & 2 \\ 7 & 1 & 3 \end{array} .$$

If $b = 7t$, one can simply take t copies of the above design. The resulting design consists of only seven distinct blocks and is therefore said to have the support size 7. There are BIB designs with different support sizes. For example if $b = 35$, the collection of all $\binom{7}{3} = 35$ possible blocks of size 3 form a BIB design; and this design has the support size 35. To the experimenter the implementation of designs with different support sizes may cost differently. On the other hand certain mixtures of treatments may be more preferrable than others. These considerations lead to the search for BIB designs with various support sizes. It is then natural to ask the following question: For $v = 7$, $k = 3$, $b = 7t$, and a given number b^*, does there exist a BIB design consisting of b^* distinct blocks?

In our setting, we may require that b^* satisfies the obvious inequalities

$$b^* \leq b \quad \text{and}$$

$$35 \geq b^* \geq 7.$$

As we shall see in Section 3, the answer to the above question is basically yes with a few exceptional cases. The construction of designs or proof of nonexistence of designs heavily relied on a method called "trade off", which is introduced and studied in the next section.

2. <u>The Method of Trade Off</u>.

Let $V = \{1,2,\ldots,v\}$. A 2-element subset of V will be called a pair and a k-element subset will be called a block (later we will concentrate on the case $k = 3$). Let P denote the incidence matrix of pairs versus blocks. So P is a $\binom{v}{2}$ by $\binom{v}{k}$ zero-one matrix. A $\binom{v}{k}$-dimensional column vector F with non-negative integer entries is called a BIB design if

$$PF = \lambda \underline{1}$$

for some positive integer λ. An entry in F represents the multiplicity (frequency) that the corresponding block appears in the design. Such a design is also called a BIB (v,b,r,k,λ)-design, where

$$r = \lambda(v-1)/(k-1) \quad \text{and} \quad b = vr/k.$$

An integer vector T of the same dimension is called a <u>trade</u> if

$$PT = \underline{0}.$$

The sum of all positive entries in a trade is called its <u>volume</u>. Let F be a BIB design. For every trade T, the vector $F + T$ is another design with the same parameters provided that all of its entries are nonnegative. Conversely, any BIB design sharing the same parameters with F can be written in the form $F + T$ for some trade T. In order to search for all designs with the same parameters as F, it then suffices to investigate the trades.

Hereafter unless specifically mentioned, we shall restrict our attention to only the case $k = 3$. So a block now means a triplet. The notation for the triplet consisting of the elements x, y and z will be (xyz), while the order among the 3 elements is not essential. Similarly, the typical notation for a pair will be (xy), $x \in V$ and $y \in V$.

Let Σ be the free **Z**-module generated by all the $\binom{v}{2}$ possible pairs and Σ_3 the free Z-module generated by all the $\binom{v}{3}$ possible triplets. The incidence matrix P in the above may now be interpreted as the <u>boundary operator</u>

$$\sigma : \Sigma_3 \to \Sigma_2,$$

which is the Z-homomorphism defined via

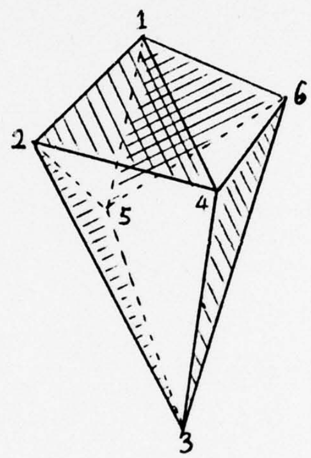
$$\sigma\,(xyz) = (xy) + (xz) + (yz).$$

Under these notations, an element in Σ_3 represents a trade if and only if it belongs to the kernel of σ.

<u>Example 1</u>. (125) + (146) + (234) + (356) - (124) - (156) - (235) - (346) represents a trade. When this trade is added to the design (124) + (137) +(156) + (235) + (267) + (346) + (457), we obtain another design (125) + (137) + (146) + (234) + (267) + (356) + (457). In other words, from the first design the four blocks (124), (156), (235), and (346) have been traded for the blocks (125), (146), (234), and (356) to obtain the second design.

Now we introduce a geometric representation of the trades. Given a trade T, construct a compact surface without boundary as follows. First create two collections of 2-simplexes (triangles) with their vertices labeled by elements of V. The 2-simplexes in one collection will be called the <u>positive triangles</u> and those in the other collection will be called the <u>negative triangles</u>. For every term +(xyz) in T, there corresponds a positive triangle with vertices labeled by x, y, and z. If the coefficient of (xyz) in T is $m > 1$, then there are m copies of such a triangle. On the other hand, for every term -(xyz) in T, there corresponds a negative triangle in the similar manner. So every pair (xy) appears on the same number of triangles in both collections. Thus, there exists a one-to-one matching between the edges of positive triangles and the edges of negative triangles so that every matched pair share the same two labels. When we identify every matched pair of edges in the natural way, we obtain a compact surface without boundary. This is because a trade is equivalent to an element in the kernel of the boundary operator. We emphasize the possible nonuniqueness of the matching. Different matchings may lead to different geometric configurations. Also the labels on the vertices are not necessarily all distinct.

<u>Example 2</u>. The trade in Example 1 is represented by the diamond-shaped topological sphere



Here in the picture the shaded regions are the negative triangles.

In general, a trade gives rise to a compact surface that is partitioned into positive triangles and negative triangles with the following two properties.

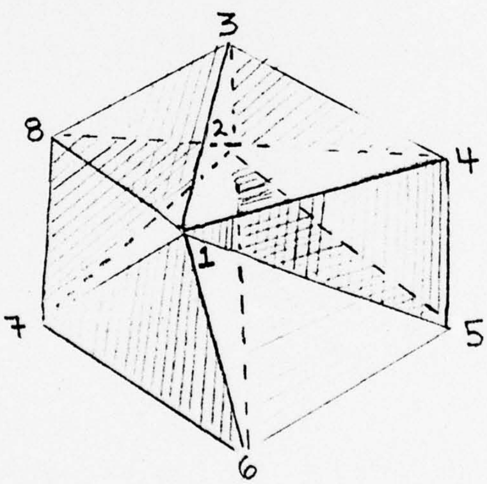
(1) Any two positive triangles can not intersect each other except possibly at their vertices. Neither can any two negative triangles.

(2) The intersection of a positive triangle with a negative

triangle is vaccum, or one vertex, or two vertices, or an edge.

We shall refer to such a partition of surfaces, with or without boundary, as an Eulerian triangulation, although it is not quite a triangulation in the usual sense of algebraic topology. The edges of the triangles form an Eulerian graph* on the surface, i.e., a graph such that the valency of every vertex is an even integer. Also no vertex can have just two valencies, because then there would be two triangles sharing two common edges.

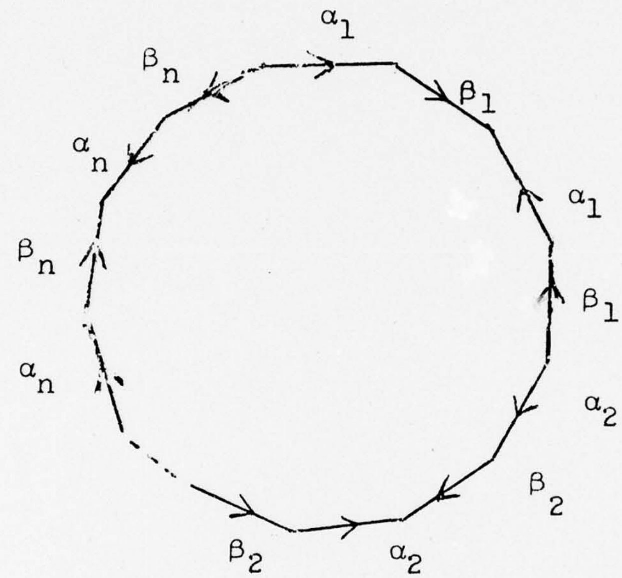
The following example of trade is also obtained by triangulating a sphere.



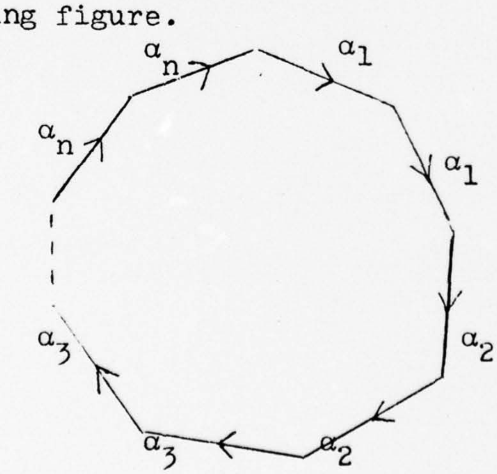
This figure represents the trade (134) + (156) + (178) + (238) + (245) + (267) - (138) - (145) - (167) - (234) - (256) - (278). Again the shaded regions are the negative triangles.

* A more precise terminology would be Eulerian multigraph than Eulerian graph according to Harary (1969).

It is well-known that a compact connected surface is either a sphere, or a connected sum of tori, or a connected sum of projective planes (see, for example, Theorem 5.1 in Massey (1967)). The standard presentation of the connected sum of n tori is by identifying edges of a $4n$-gon in pairs.

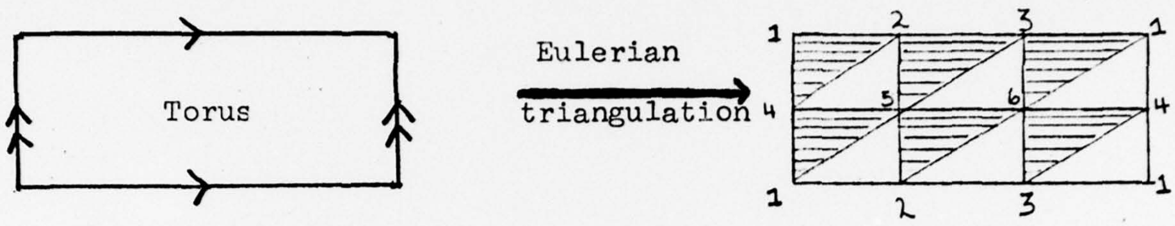


Similarly for the connected sum of n projective planes we have the following figure.

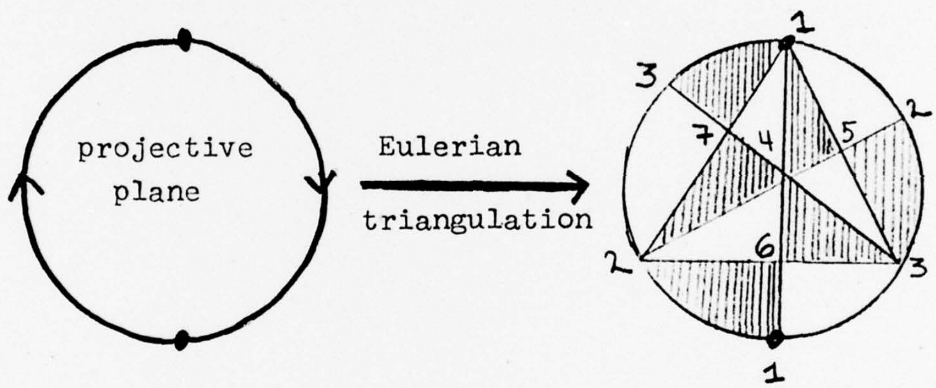


Using these standard presentation of surfaces, we can easily construct more trades.

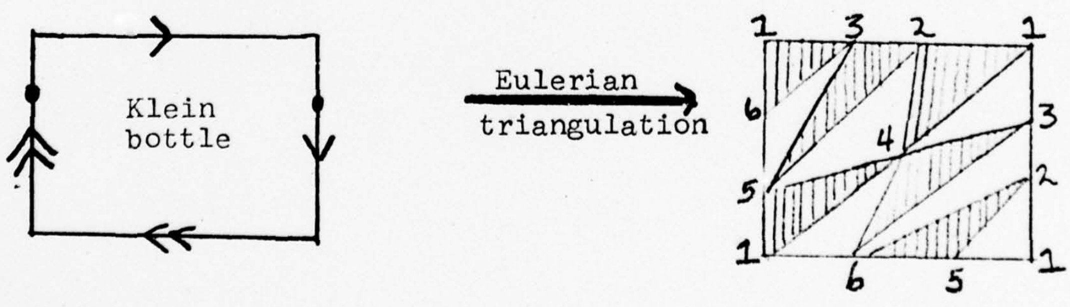
Example 4.



Example 5.

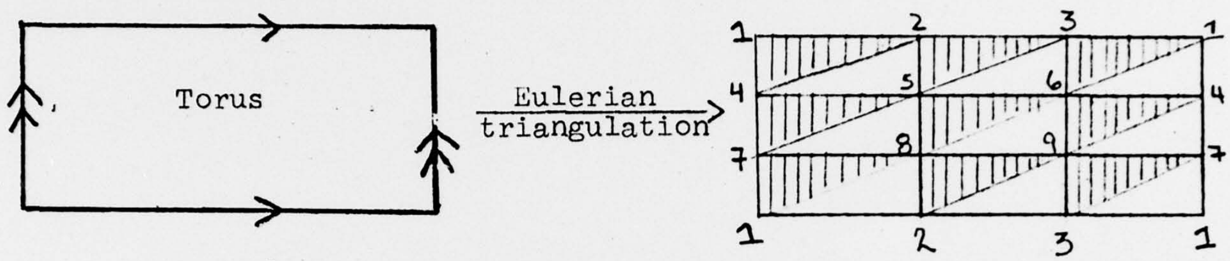


Example 6.

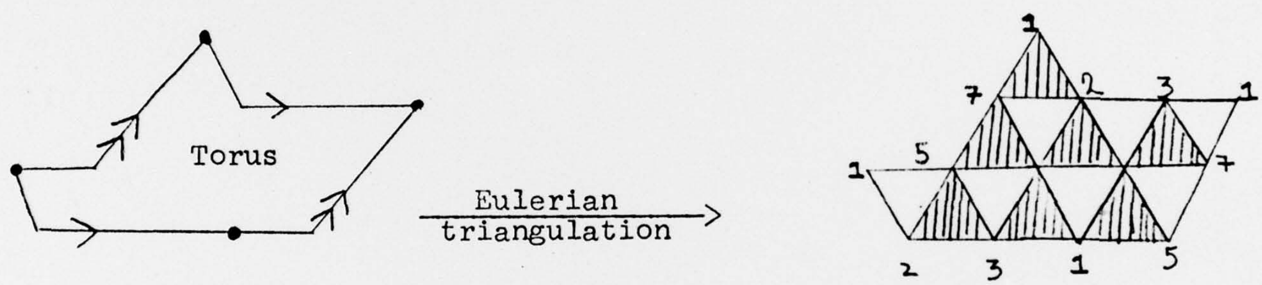


Note that the figures in Examples 4 and 6 represent the same trade.

Example 7.



Example 8.



Recall that the volume of a trade (as a column vector) means the total positive entry in the trade. We have the following:

Lemma 1. For a sufficiently large v, there exist trades of any volume other than 1, 2, 3, and 5.

Proof: In Examples 2, 4, 8, and 7, we have seen trades of volume 4, 6, 7, and 9, respectively. On the other hand, adding m copies of a trade of volume 4 to a trade of volume k based on unrelated symbols yields a trade of volume $4m + k$. This observation together with the above examples proves the lemma.

We have seen the convenience in constructing trades from the concept of Eulerian triangulation. In the proof of Lemma 5 below, we shall also find the same concept powerful in showing negative results. First we state a couple of self-evident lemmas.

Lemma 2. For every Eulerian triangulation of a compact surface with boundary, the number of boundary edges that are on positive triangles differs from the number of those on negative triangles by a multiple of 3.

Lemma 3. There exist no trades of volume 1, 2, *or* 3; *therefore the minimum trade volume is* 4.

Lemma 4. If a disc is Eulerian triangulated with exactly 2 *boundary edges, then*

(i) *exactly one boundary edge is on a positive triangle and the other is on a negative triangle, and*

(ii) *there are at least* 4 *positive and* 4 *negative triangles*.

Proof: Statement (i) follows directly from Lemma 2. From this, we know the Eulerian triangulation represents a trade, even though the surface has a boundary. The second statement now follows from Lemma 3.

Lemma 5. There exist no trades of volume 5.

Proof: Assuming there exists an Eulerian triangulation of certain compact surface without boundary by exactly 5 positive and 5 negative triangles, we want to derive a contradiction. First, we know that the triangulation on every connected component of the surface represents a trade. So the surface must be connected by Lemma 3. There are 10 triangles in total, so there are 15 edges. Let n be the number of vertices. The Euler characteristic of this surface is

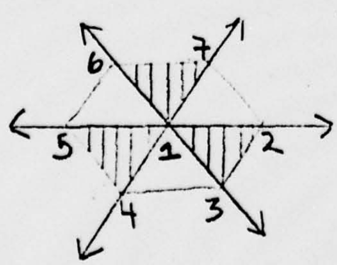
$$\begin{aligned}\chi &= n - 15 + 10 \\ &= n - 5 \\ &\leq 2.\end{aligned}$$

The inequality has been due to the connectedness. We label the vertices by $1, 2, \ldots, n$, respectively. There are three cases to examine.

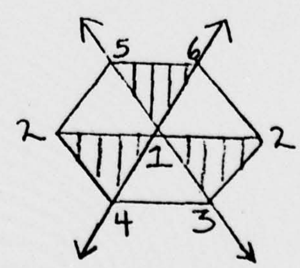
Case 1. $\chi = 2$. Then $n = 7$ and the surface is a topological sphere. The edges in the triangulation form a planar graph and its valency sequence is

$$(6, 4, 4, 4, 4, 4, 4,).$$

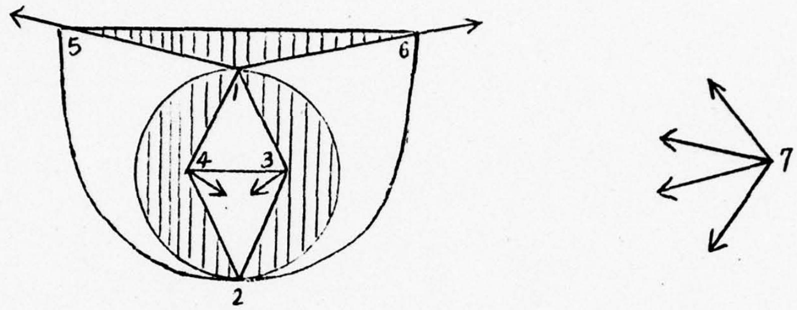
With a suitable relabeling, the neighborhood around the vertex is as in either graph below.



or



In the first graph, the six arrows are supposed to be linked in pairs to form a planar graph, but this is obviously impossible. After identifying the two points labeled as 2, the second graph lead to the following configuration.



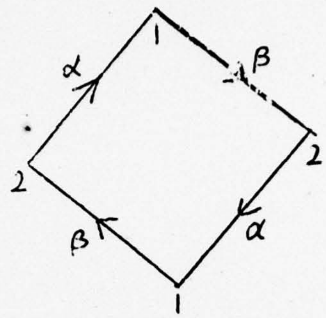
Again the arrows can not be linked in pairs to form a planar graph.

Case 2. $\chi = 1$. Then $n = 6$ and the surface is a projective plane. The valency sequence has to be one of the following three:

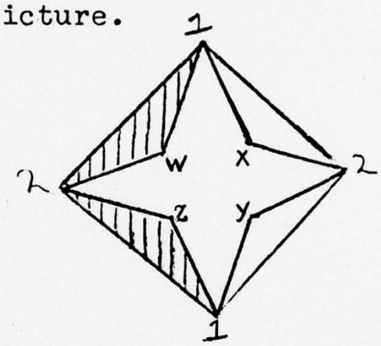
$$(6, 6, 6, 4, 4, 4)$$
$$\text{or } (8, 6, 4, 4, 4, 4)$$
$$\text{or } (10, 4, 4, 4, 4, 4).$$

Since in any case some vertex has at least 6 valencies, we may assume that there are two edges α and β joining between vertices 1 and 2. These two edges form a cycle. Since the fundamental group of a projective plane is $Z/2Z$, this cycle is either trivial or is the generator of the fundamental group.

First we assume that the cycle generates the fundamental group. Then the projective plane can be drawn as a square with edges identified in pairs as in below.



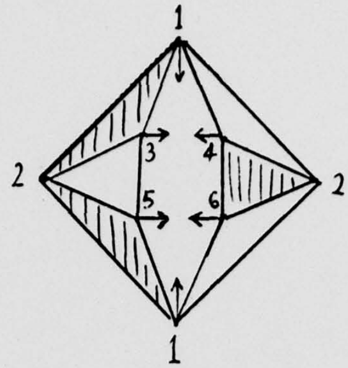
So we have an Eulerian triangulation of the square disc based on the following picture.



Here $w, x, y, z \in \{3,4,5,6\}$ and $w \neq x$, $w \neq z$, $x \neq y$, $y \neq z$. Also from Lemma 4, we have $w \neq y$ and $x \neq z$. So w, x, y, and z are all distinct. By symmetry, let $w = 3$, $x = 4$, $y = 5$, and $z = 6$. Observe that vertex 1 must have more than 6 valencies and vertex 2 has at least 6. Therefore the valency sequence is

$$(8, 6, 4, 4, 4, 4),$$

and the arrows in the following graph should be linked in pairs to form the triangulation.



But [illegible] is obviously impossible.

[illegible]ssume that α and β form a trivial cycle. The [illegible] cuts the projective plane into two parts: a disc and a Möbius band. From Lemma 4 the Eulerian triangulation on the disc part takes at least 4 positive and 4 negative triangles. So the Möbius band is Eulerian triangulated by at most 1 positive and 1 negative triangles. This is a contradiction.

Case 3. $\chi = 0$. Then $n = 5$. We need to show the nonexistence of a trade T of volume 5 on 5 or less symbols. First we may assume that T is of the form

$$(123) - (124) - (134) - (23x) + - \ldots,$$

where $x = 4$ or 5. Then the coefficient of the block (145) in T must be at least 2. Thus

$$\begin{aligned} T = {} & (123) + 2(145) - (124) - (134) - (23x) \\ & - (1y5) - (1z5) - (u45) - (v45) + - \ldots \end{aligned}$$

But this implies that T has volume at least 7, a contradiction.

We conclude the above results in the following theorem for later application.

THEOREM 1. *For any integer* i, *there exists a trade of volume* i *if and only if* $i \neq 1, 2, 3,$ *or* 5.

Remark: It is natural to generalize the concept of Eulerian triangulation for higher dimensional manifolds. Then a t-dimensional Eulerian triangulation represents a trade on the so-called *t-designs* with $k = t + 1$. BIB designs are t-designs when $t = 2$. Since only the 2-dimensional compact manifolds have been completely classified, the analysis of higher dimensional Eulerian triangulations may be difficult.

3. *An Application of the Method of Trade Off: BIB designs with* $v = 7$ *and* $k = 3$.

All the *designs* in this section refer to BIB$(7,b,r,3,\lambda)$-designs based on the set of symbols $\{1,2,3,4,5,6,7\}$. From the relations $rv = bk$ and $\lambda(v-1) = r(k-1)$, one can see that b must be a multiple of 7. Also, we have $r = 3b/7$ and $\lambda = b/7$. Thus there are b blocks in the design; every symbol appears in exactly r of them and every pair occurs in exactly λ of them. We want to determine the existence or nonexistence of designs with any given b and b^* (the support size). The results are summarized in Table 1.

Table 1.
Existence and nonexistence of BIB(7,b,r,3,λ)-designs with support size b*. The symbol "X" indicates nonexistence. A blank space means existence.

b \ b*	6 or less	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
7	X		Nonexistence (because b* is never greater than b)																											
14	X		X	X	X		X																							
21	X		X	X	X		X				Existence is doubtful																			
28	X		X	X	X		X															X								
35	X		X	X	X		X																		X		X	X	X	
42	X		X	X	X		X																						X	
49 or more	X		X	X	X		X																							

From Table 1 the only unknown case is when $b^* = 16$. In all other cases we shall either exhibit examples of designs with prescribed b and b^* or prove their nonexistence.

First, in order to cover all the $\binom{7}{2} = 21$ possible pairs, at least 7 distinct blocks are needed in a design. This means that $b^* \geq 7$. In particular, it is known that every design with $b = b^* = 7$ is isomorphic to a finite projective plane of order 2.

Theorem 3.2 in van Lint and Ryser (1972) shows b^* can never be 8. Pesotchinsky (1977) showed $b^* \neq 9, 10, 12$.

When $b = 35$, there are no designs based on exactly 30, 32, 33, or 34 distinct blocks. Because if there existed such a design, its difference from the complete design would be a trade of volume 5, 3, 2, or 1, contradicting Theorem 1 in the last section.

Next we treat the case when $(b,b^*) = (28,27)$ or $(42,34)$. We need the following lemma.

Lemma 6. *Let* F *be a design with* $b = 14$, *and* $B_1, B_2, \ldots, B_8$ *be blocks. If* $F + (123) - B_1 - B_2 - \ldots - B_8$ *is a trade, then* $B_i = (123)$ *for some* i.

Proof: Assuming that $B_i \neq (123)$ for all i, we shall derive a contradiction. The blocks $B_1, B_2, \ldots, B_8$ cover all the 21 possible pairs; the three pairs (12), (13) and (23) are doubly covered, while all other pairs are singly covered. By symmetry, we may assume that $B_1 = (12u)$, $B_2 = (12v)$, $B_3 = (13w)$, $B_4 = (13x)$, $B_5 = (23y)$, and $B_6 = (23z)$. But the above covering properties of the eight blocks imply that u, v, w, x, y, z are distinct elements of the 4-element set $\{4,5,6,7\}$. This is a contradiction.

Proposition 1. $(b,b^*) \neq (28,27)$.

Proof: Suppose there exists a design F with $b = 28$ and $b^* = 27$. We may assume, by symmetry, that the unique doubled block in the design is (123). Let the 8 blocks that are missing from the design be denoted as $B_1, B_2, \ldots, B_8$. Thus $B_i \neq (123)$ for all i. Adding any design with 7 blocks to F minus the complete design, we obtain a trade that is prohibited by the above lemma. This contradiction originates from the assumption of the existence of the design F.

Proposition 2. $(b,b^*) \neq (42,34)$.

Proof: Suppose there exists a design with $b = 42$ and $b^* = 34$. By symmetry, we assume that this design is equal to

$$\text{the complete design} - (123) + B_1 + B_2 + \ldots + B_8,$$

where $B_1, B_2, \ldots, B_8$ are blocks not equal to (123). The proof now proceeds in the same way as in the proof of Proposition 1.

As a contrast against Lemma 6, we have the following:

Example 9: Let S denote the sum

$(124) + (126) + (127) + (134) + (135) + (137) + (156) + (234) + (235) + (236) + (257) + (367) + (456) + (457) + (467) - (123)$. If F is any design with 14 blocks, then $F - S$ is a trade. In fact, up to isomorphism, S is unique subject to this property. On the other hand, the complete design plus S is a design with $b = 49$ and $b^* = 34$.

In Table 2 through Table 6, we provide examples of designs with various b and b^*. In these tables each column specifies a BIB design. Each entry indicates the number of copies of the corresponding block in the design.

Table 2
BIB Designs With v = 7 and k = 3
All Possible Support Sizes When b = 14

blocks \ b*	7	8	9	10	11	12	13	14
123	-				-		-	-
124	2				1		-	1
125	-				-		1	-
126	-				-		-	1
127	-				1		1	-
134	-				-		-	1
135	2				-		-	1
136	-				2		2	-
137	-				-		-	-
145	-				1		1	-
146	-				-		-	-
147	-				-		1	-
156	-				-		-	-
157	-				1		-	1
167	2				-		-	1
234	-				1		1	-
235	-				-		-	1
236	-				-		-	-
237	2	DOES NOT EXIST	DOES NOT EXIST	DOES NOT EXIST	1	DOES NOT EXIST	1	1
245	-				-		-	-
246	-				-		1	-
247	-				-		-	1
256	2				2		1	1
257	-				-		-	-
267	-				-		-	-
345	-				1		1	-
346	2				-		-	1
347	-				-		-	-
356	-				-		-	-
357	-				1		1	-
367	-				-		-	1
456	-				-		-	1
457	2				-		-	1
467	-				2		1	-
567	-				-		1	-

Table 3
BIB Designs With v = 7 and k = 3
All Possible Support Sizes When b = 21

blocks \ b*	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
123	-				2		2	-	-		-	1	-	1	1
124	-				1		1	1	-		-	1	3	1	1
125	-				-		-	-	1		1	1	-	1	1
126	3				-		-	2	-		-	-	-	-	-
127	-				-		-	-	2		2	-	-	-	-
134	3				-		-	2	2		1	-	-	-	-
135	-				-		1	-	-		-	1	1	1	-
136	-				1		-	-	1		2	1	1	-	1
137	-				-		-	1	-		-	-	1	1	1
145	-				-		2	-	1		1	1	-	1	1
146	-				2		-	-	-		-	-	-	1	-
147	-				-		-	-	-		1	1	-	-	1
156	-				-		-	1	1		1	-	1	-	1
157	3				3		-	2	-		-	-	1	-	-
167	-				-		3	-	1		-	2	1	2	1
234	-				-		-	-	1		1	1	-	1	1
235	-	DOES NOT EXIST	DOES NOT EXIST	DOES NOT EXIST	1	DOES NOT EXIST	-	1	-	ITS EXISTENCE IS DOUBTFUL	-	-	1	-	-
236	-				-		1	-	2		1	1	1	1	-
237	3				-		-	2	-		1	-	1	-	1
245	3				2		-	2	1		1	-	-	-	-
246	-				-		-	-	-		1	-	-	-	1
247	-				-		2	-	1		-	1	-	1	-
256	-				-		2	-	1		1	1	1	1	1
257	-				-		1	-	-		-	1	1	1	1
267	-				3		-	1	-		-	1	1	1	1
345	-				-		-	-	-		1	-	1	-	1
346	-				-		2	1	-		-	1	1	1	1
347	-				3		1	-	-		-	1	1	1	-
356	3				2		-	2	-		-	-	-	1	1
357	-				-		2	-	3		2	2	-	1	1
367	-				-		-	-	-		-	-	-	-	-
456	-				1		1	-	1		-	2	1	1	-
457	-				-		-	1	-		-	-	1	1	1
467	3				-		-	2	2		2	-	1	-	1
567	-				-		-	-	-		1	-	-	-	-

Table 4
BIB Designs With v = 7 and k = 3
All Possible Support Sizes When b = 28

blocks \ b*	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
123	-	DOES NOT EXIST	DOES NOT EXIST	DOES NOT EXIST	2	DOES NOT EXIST	-	3	-	ITS EXISTENCE IS DOUBTFUL	1	-	1	2	1	-	-	-	-	1	DOES NOT EXIST	1
124	4				2		-	-	-		-	-	1	1	1	-	3	2	2	1		1
125	-				-		4	1	2		3	3	2	1	2	2	-	1	1	1		1
126	-				-		-	-	-		-	-	-	-	-	-	1	-	-	-		-
127	-				-		-	-	2		-	1	-	-	-	2	-	1	1	1		1
134	-				-		1	1	3		1	1	1	-	-	1	1	-	-	-		-
135	4				-		-	-	-		-	-	1	-	1	1	1	2	2	1		1
136	-				2		3	-	1		2	3	2	2	2	2	1	1	1	1		1
137	-				-		-	-	-		-	-	-	-	1	-	1	1	1	1		1
145	-				-		-	-	1		-	1	-	1	-	1	-	-	-	-		1
146	-				2		-	-	-		-	-	-	-	1	1	-	1	1	2		1
147	-				-		3	3	-		3	2	2	2	2	1	-	1	1	1		1
156	-				-		-	3	1		1	-	-	1	-	-	1	1	1	1		1
157	-				4		-	-	-		-	-	-	1	-	-	2	-	-	1		-
167	4				-		1	1	2		1	1	2	1	1	1	1	1	1	-		1
234	-				-		-	-	1		-	1	1	1	1	1	-	1	2	1		1
235	-				2		-	-	-		-	-	1	-	1	-	2	1	-	-		-
236	4				-		1	1	3		1	1	-	-	-	2	1	1	1	1		1
237	-				-		3	-	-		2	2	2	1	2	1	1	1	1	1		1
245	-				2		-	-	1		-	-	-	1	-	1	-	1	1	1		1
246	-				-		3	3	-		3	2	1	-	1	1	-	1	-	1		1
247	-				-		1	1	2		1	1	-	1	-	1	1	-	-	-		-
256	-				-		-	-	1		-	1	2	2	2	1	1	-	1	1		1
257	4				-		-	3	-		1	-	-	-	-	-	1	1	1	1		1
267	-				4		-	-	-		-	-	2	2	2	-	1	1	1	1		1
345	-				-		3	3	-		3	2	-	1	1	1	1	1	1	2		1
346	-				-		-	-	-		-	-	1	2	1	-	1	1	1	-		1
347	4				4		-	-	-		-	-	-	-	-	1	1	2	1	1		1
356	-				2		-	-	-		-	-	1	-	1	-	-	-	-	1		1
357	-				-		1	1	4		1	2	1	3	-	2	-	-	1	-		1
367	-				-		-	3	-		1	-	-	-	-	-	1	-	-	1		-
456	4				2		1	1	2		1	1	2	1	1	1	2	1	1	-		-
457	-				-		-	-	-		-	-	2	-	2	-	1	1	1	1		1
467	-				-		-	-	2		-	1	-	1	-	1	1	-	1	1		1
567	-				-		3	-	-		2	2	-	-	1	2	-	2	1	1		1

Table 5
BIB Designs With v = 7 and k = 3
All Possible Support Sizes When b = 35

blocks \ b*	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
123	5	DOES NOT EXIST	DOES NOT EXIST	DOES NOT EXIST	5	DOES NOT EXIST	-	-	5	ITS EXISTENCE IS DOUBTFUL	-	-	-	-	4	-	3	1	-	-	1	2	-	DOES NOT EXIST	1	DOES NOT EXIST	DOES NOT EXIST	DOES NOT EXIST	1
124	-				-		-	1	-		2	1	3	3	1	-	2	1	2	2	2	2	1		1				1
125	-				-		1	-	-		-	1	-	1	-	3	-	1	1	1	-	1	1		1				1
126	-				-		4	4	-		3	3	1	1	-	1	-	2	2	2	1	-	2		2				1
127	-				-		-	-	-		-	-	1	-	-	1	-	-	-	-	1	-	1		-				1
134	-				-		-	4	-		3	-	-	-	-	1	-	-	2	1	1	1	1		1				1
135	-				-		-	1	-		1	-	-	2	1	-	2	1	1	1	-	1	1		1				1
136	-				-		-	-	-		1	1	4	3	-	2	-	-	1	2	2	1	2		1				1
137	-				-		5	-	-		-	4	1	-	-	2	-	3	1	1	1	-	1		1				1
145	-				-		4	-	-		-	3	2	-	-	2	-	2	1	1	1	-	2		1				1
146	5				4		1	-	4		-	1	-	-	3	1	2	1	-	-	1	1	-		1				1
147	-				1		-	-	1		-	-	-	2	1	1	1	1	-	1	-	1	1		1				1
156	-				1		-	-	1		-	-	-	-	1	-	1	1	-	-	1	1	-		-				1
157	5				4		-	4	4		4	1	3	2	3	-	2	-	2	2	3	2	1		2				1
167	-				-		-	1	-		1	-	-	1	1	1	2	1	2	1	-	2	1		1				1
234	-				-		1	-	-		-	1	2	2	-	1	-	3	1	2	1	-	2		1				1
235	-				-		4	4	-		4	4	1	2	-	2	-	1	2	2	1	-	2		1				1
236	-				-		-	-	-		-	-	-	-	1	-	1	-	-	-	1	1	-		-				1
237	-				-		-	1	-		1	-	2	1	-	2	1	-	2	1	1	2	1		2				1
245	5				5		-	-	4		-	-	-	-	3	-	2	-	-	-	1	1	-		1				1
246	-				-		-	-	-		-	-	-	-	-	3	-	-	1	-	-	1	1		1				1
247	-				-		4	4	1		3	3	-	-	1	1	1	1	1	1	1	1	1		1				1
256	-				-		-	1	1		1	-	3	1	1	-	2	1	1	1	2	2	1		1				1
257	-				-		-	-	-		-	-	1	1	1	-	1	2	1	1	1	1	1		1				1
267	5				5		1	-	4		1	2	1	3	3	1	2	2	1	2	1	1	1		1				1
345	-				-		-	-	1		-	-	2	-	1	1	1	-	-	-	2	2	-		1				1
346	-				1		4	1	1		1	3	-	1	1	1	2	2	1	1	-	1	1		1				1
347	5				4		-	-	3		1	1	1	2	3	1	2	-	1	1	1	1	1		1				1
356	5				4		1	-	3		-	1	1	-	2	2	1	2	2	1	1	1	1		2				1
357	-				1		-	-	1		-	-	1	1	1	-	1	1	-	1	1	1	1		-				1
367	-				-		-	4	1		3	-	-	1	1	-	1	1	1	1	1	1	1		1				1
456	-				-		-	4	-		4	1	1	4	1	-	1	1	2	3	1	1	2		1				1
457	-				-		1	1	-		1	1	-	1	-	2	1	2	2	1	-	1	1		1				1
467	-				-		-	-	-		-	-	4	-	-	-	-	1	1	1	3	1	1		1				1
567	-				-		4	-	-		-	3	-	-	-	3	-	-	-	-	-	-	1		1				1

Table 6
BIB Designs With $v = 7$, $k = 3$, $b = 42$
and Support Sizes 30, 32, 33

blocks \ b*	30	32	33
123	1	1	1
124	1	1	1
125	1	1	1
126	2	2	2
127	1	1	1
134	1	1	2
135	1	2	1
136	2	1	1
137	1	1	1
145	3	1	1
146	-	2	1
147	1	1	1
156	-	-	1
157	1	2	2
167	2	1	1
234	2	2	1
235	2	1	2
236	-	-	-
237	1	2	2
245	-	1	1
246	2	1	2
247	1	1	1
256	1	2	1
257	2	1	1
267	1	1	1
345	-	1	1
346	1	1	1
347	2	1	1
356	2	2	2
357	1	-	-
367	1	2	2
456	2	1	1
457	1	2	2
467	1	1	1
567	1	1	1

With the designs in these tables and the one constructed at the end of Example 9, we need only to observe the following fact in order to establish all the existences claimed in Table 1.

<u>Proposition 3</u>. <u>If there exists a design with b blocks which contains a design with 7 blocks, then there exists a design with $b + 7$ blocks and the same support size</u>.

We now explain by an example the way the designs in the above tables were obtained through the method of trade off.

<u>Example 10</u>. Let F denote the design (127) + (134) + (156) + (235) + (246) + (367) + (457) and T the trade (127) + (156) + (236) + (357) - (126) - (157) - (237) - (356). Then the complete design plus F minus T is a design with 42 blocks. Since only the two blocks (236) and (357) are missing from this design, the support size is 33. This is the design shown on the last column in Table 6.

<u>Open Question</u>: Does there exist any design with the support size $b^* = 16$ (while b can be any number)?

The authors conjecture the negative answer to this question. The conjecture is based on the following, hopefully plausible, argument. If there exist any designs with $b^* = 16$, then very likely there exist such designs with $b = 21$. It can be shown that a design with $(b,b^*) = (21,16)$ can not contain a sub BIB design based on 7 blocks. However computer search suggests that any design with $b^* < 21$ should contain a sub BIB design based on 7 blocks. So the existence of designs with $b^* = 16$ is doubtful. Incidentally, whether the computer search has suggested a fact is another open question.

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